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LARGE DEVIATIONS FOR PAST-DEPENDENT RECURSIONS

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ABSTRACT. The Large Deviation Principle is established for stochastic models defined by past-dependent non linear recursions with small noise. In the Markov case we use the result to obtain an explicit expression for the asymptotics of exit time.

1. Introduction.

The simplest example of a stochastic model defined by past-dependent recursion with small noise is a linear model

$$X_k^\varepsilon = \sum_{i=1}^m a_i X_{k-i}^\varepsilon + \varepsilon \xi_k \quad (1.1)$$

subject to fixed $X_i^\varepsilon = x_i, i = 0, 1, \dots, m-1$, where ε is a small parameter and $(\xi_k)_{k \geq m}$ is an i.i.d. sequence of random variables. In the present paper we consider a general non linear model:

$$X_k^\varepsilon = f(X_{k-1}^\varepsilon, \dots, X_{k-m}^\varepsilon, \varepsilon \xi_k), \quad (1.2)$$

where $f(z_1, \dots, z_m, y)$ is a continuous function. Note that the model (1.2) includes (1.1) as a special case. For $m = 1$ the model (1.2) defines a discrete time Markov process. When $\varepsilon \rightarrow 0$ random variables X_k^ε converge to deterministic ones, say, X_k and $X_k, k \geq 1$ are determined by the recursion

$$X_k = f(X_{k-1}, \dots, X_{k-m}, 0)$$

subject to the same initial condition. Furthermore $(X_k^\varepsilon)_{k \geq m}$ converges to $(X_k)_{k \geq m}$ in the metric $\rho(x, y) = \sum_{j \geq m} 2^{-j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$. This fact provides the motivation to consider the large deviation principle (LDP) for family $(X_k^\varepsilon)_{k \geq m}$ in the metric space $(\mathbb{R}^\infty, \rho)$. For Markov case ($m = 1$) the LDP was considered in [5], [7] and [8]. The choice of the metric space $(\mathbb{R}^\infty, \rho)$ is a natural one for obtaining the LDP for the family $(X_k^\varepsilon)_{k \geq m}$. Recursion (1.2) defines continuous mapping $(\varepsilon \xi_k)_{k \geq m} \rightarrow (X_k^\varepsilon)_{k \geq m}$ in the metric ρ . This implies that the LDP for $(X_k^\varepsilon)_{k \geq m}$ follows from the LDP for $(\varepsilon \xi_k)_{k \geq m}$ by the continuous mapping method of Freidlin [3] or contraction principle of Varadhan [11]. Since $(\xi_k)_{k \geq m}$ is an i.i.d. sequence, the LDP for $(\varepsilon \xi_k)_{k \geq m}$ holds if it holds for the family $\varepsilon \xi$, where ξ is a copy of ξ_k . It should be mentioned that not only the rate function but also the rate of speed $q(\varepsilon)$ depends on the distribution of ξ .

Key words and phrases. Large Deviations, Contraction Principle, Exit Time.

In Section 4, sufficient conditions proving the LDP for family $\varepsilon\xi$ are given. Section 3 contains examples for which rate functions can be explicitly calculated, and in the Markov case asymptotics for the probability of $\{\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1\}$ is found (Theorem 3.1). Main results are formulated in Section 2. One of them gives the asymptotics of the exit time from the interval $[-1, 1]$ for the Markov family $(X_k^\varepsilon)_{k \geq 1}$.

2. Main results

Following Varadhan [11], family $(X^\varepsilon)_{k \geq m}$ is said to satisfy the LDP in the metric space $(\mathbb{R}^\infty, \rho)$ with the rate of speed $q(\varepsilon)$ and the rate function $J(u)$ if

(0) there exists a function $J = J(\bar{u})$, $\bar{u} = (u_1, u_2, \dots) \in \mathbb{R}^\infty$ which takes values in $[0, \infty]$ such that for every $\alpha \geq 0$ the set $\Phi(\alpha) = \{\bar{u} \in \mathbb{R}^\infty : J(\bar{u}) \leq \alpha\}$ is compact in $(\mathbb{R}^\infty, \rho)$;

(1) For every closed set $F \in (\mathbb{R}^\infty, \rho)$

$$\overline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log P((X_k^\varepsilon)_{k \geq m} \in F) \leq - \inf_{\bar{u} \in F} J(\bar{u});$$

(2) For every open set $G \in (\mathbb{R}^\infty, \rho)$

$$\underline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log P((X_k^\varepsilon)_{k \geq m} \in G) \geq - \inf_{\bar{u} \in G} J(\bar{u}).$$

As was mentioned in Introduction, the LDP for $(X_k^\varepsilon)_{k \geq m}$ is implied by the LDP for family $\varepsilon\xi$ (ξ is a copy of ξ_k). Therefore, we begin with the LDP $\varepsilon\xi$. Henceforth the following conditions are assumed to be fulfilled:

(A.1)

- $E\xi = 0$

- $Ee^{t\xi} < \infty$, $t \in \mathbb{R}$ “Cramer’s condition”.

(A.2) With a cumulant function $H(t) = \log Ee^{t\xi}$ and the Fenchel-Legendre transform $L(v) = \sup_{t \in \mathbb{R}} [tv - H(t)]$, there exist a function $q(\varepsilon)$, decreasing to 0 as $\varepsilon \downarrow 0$, and a nonnegative function $I(v) = \lim_{\varepsilon \rightarrow 0} q(\varepsilon)L(v/\varepsilon)$, $v \in \mathbb{R}$ with properties:

- $I(0) = 0$

- $\lim_{|v| \rightarrow \infty} I(v) = \infty$.

(A.3) If $I(v) < \infty$ for some v , then $t_v^\varepsilon = \operatorname{argmax} (t \frac{v}{\varepsilon} - H(t))$ is finite and

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{q(\varepsilon)}{\varepsilon} |t_v^\varepsilon| < \infty \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^2 H''(t_v^\varepsilon) = 0.$$

We notice also that the Cramer condition implies $H'(0) = 0$ and $H''(t) \geq 0$ and left continuity of $I(v)$ in a vicinity of $v_0 = \inf\{v > 0 : I(v) = \infty\}$ (correspondingly, right continuity for $v_0 < 0$).

The main results are given

Theorem 2.1. *Assume (A.1) - (A.3). Then:*

1) *the family $\{\varepsilon\xi\}_{\varepsilon \rightarrow 0}$ obeys the LDP with the rate of speed $q(\varepsilon)$ and the function $I(v)$ defined in (A.2);*

2) *the family $\{(\varepsilon\xi_k)_{k \geq 1}\}_{\varepsilon \rightarrow 0}$ obeys the LDP in the metric space $(\mathbb{R}^\infty, \rho)$ with*

the rate function $(\bar{v} = (v_1, v_2, \dots) \in \mathbb{R}^\infty)$

$$I_\infty(\bar{v}) = \sum_{k=1}^{\infty} I(v_k); \quad (2.1)$$

3) the family $\{(X_k^\varepsilon)_{k \geq m}\}_{\varepsilon \rightarrow 0}$ obeys the LDP in the metric space $(\mathbb{R}^\infty, \rho)$ with the rate function $(\bar{u} = (u_m, u_{m+1}, \dots) \in \mathbb{R}^\infty)$

$$J_\infty(\bar{u}) = \begin{cases} \sum_{k=m}^{\infty} \inf_{v_k: u_k = f(u_{k-1}, \dots, u_{k-m}, v_k)} I(v_k), & u_i = x_i, i = 0, \dots, m-1 \\ \infty, & \text{otherwise,} \end{cases}$$

where $\inf(\emptyset) = \infty$.

Remark 1. Some time (A.1) - (A.3) might be readily verified if ξ obeys a decomposition $\xi = \xi^i + \xi^{ii}$ with independent random summands satisfying the Cramer condition. If for ξ^i the Theorem conditions are satisfied (with $q^i(\varepsilon)$, $I^i(v)$) and

$$\lim_{\varepsilon \rightarrow 0} q^i(\varepsilon) L^{ii}(v/\varepsilon) = -\infty, \quad v \neq 0, \quad (2.2)$$

then the theorem statement is valid the rate of speed $q(\varepsilon) = q^i(\varepsilon)$ and the rate function $I(v) \equiv I^i(v)$.

Condition (2.2) always holds for random variables with a finite support.

Remark 2. The requirement for $f(z_1, \dots, z_m, y)$ to be continuous can be relaxed if

$$H_\varepsilon(t, z_1, \dots, z_m) = \log \mathbf{E} \exp \left(t f(z_1, \dots, z_m, \varepsilon \xi_1) \right)$$

is continuous in z_1, \dots, z_m for every fixed t and ε , and there exists a norming factor $q(\varepsilon)$, that is,

$$\lim_{\varepsilon \rightarrow 0} q(\varepsilon) \sup_{t \in \mathbb{R}} [tu - H_\varepsilon(z_1, \dots, z_m)] = I(u, z_1, \dots, z_m).$$

Then, the LDP for the family $\{(X_k^\varepsilon)_{k \geq m}\}_{\varepsilon \rightarrow 0}$ may hold with rate function

$$J_\infty(u_1, u_2, \dots) = \sum_{k \geq m} I(u_k, u_{k-1}, \dots, u_{k-m}).$$

The asymptotics for exit time is our next result. Let

$$X_k^\varepsilon = aX_{k-1}^\varepsilon + \varepsilon \xi_k$$

subject to $X_0^\varepsilon = 0$ and ξ_1 is $(0, 1)$ -Gaussian random variable. Denote τ^ε exit time from the interval $[-1, 1]$,

$$\tau^\varepsilon = \min\{k \geq 1 : |X_k^\varepsilon| \geq 1\}.$$

Theorem 2.2. If $|a| < 1$, then $\overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbf{E} \tau^\varepsilon \leq \frac{1}{2}(1 - a^2)$.

Remark 3. The statement from 1) related to a discrete time version of the Freidlin-Wentzell result on of the exit time asymptotics for diffusion processes [4] while a corresponding discrete time version can be found in Kifer, [6]. Unfortunately, we could not apply Kifer's result since in [6] X_k^ε take values in a compact while in our case $X_k^\varepsilon \in \mathbb{R}$. Therefore, repeating some details from [6], we give a self-contained proof.

3. Examples and Applications.

Example 3.1. The rate of speed $q(\varepsilon) = \varepsilon^2$ and the rate function $I(v) = \frac{1}{2}v^2$ correspond to the family $\{\varepsilon\xi\}_{\varepsilon \rightarrow 0}$ with $(0, 1)$ -Gaussian random variable with the cumulant function $H(t) = \frac{t^2}{2}$. At the same time the pair $q(\varepsilon) = \varepsilon|\log \varepsilon|$, $I(v) = |v|$ correspond to the Poisson random variable ξ with parameter 1 and the cumulant function $H(t) = e^t + e^{-t} - 2$.

It is interesting to note that for $\xi = \xi^i + \xi^{ii}$, where ξ^i and ξ^{ii} are independent random variables:

- ξ^i is the Gaussian(0, 1) random variable,
- ξ^{ii} is the Poisson(1) random variable,

then the LDP for family $\{\varepsilon\xi\}_{\varepsilon \rightarrow 0}$ holds with $q(\varepsilon) = \frac{\varepsilon}{|\log \varepsilon|}$ and $I(v) = |v|$.

Example 3.2. For a linear in y function $f(z_1, \dots, z_m, y)$, involving in (1.2):

$$f(z_1, \dots, z_m, y) = a(z_1, \dots, z_m) + b(z_1, \dots, z_m)y,$$

with positive $b(z_1, \dots, z_m)$, and Gaussian(0, 1) random variable ξ_1 the rate function is defined as:

$$J_\infty(\bar{u}) = \begin{cases} \sum_{k=m}^{\infty} \frac{(u_k - a(u_{k-1}, \dots, u_{k-m}))^2}{b^2(u_{k-1}, \dots, u_{k-m})}, & u_0 = x_0, \dots, u_{m-1} = x_{m-1} \\ \infty, & \text{otherwise.} \end{cases}$$

It can be shown, in particular, that the above formula for the rate function is preserved if $b(z_1, \dots, z_m)$ equals zero for some (z_1, \dots, z_m) provided the convention $0/0 = 0$.

In the case of the Markov model $X_k^\varepsilon = a(X_{k-1}^\varepsilon) + b(X_{k-1}^\varepsilon)\varepsilon\xi_k$, an analogy to Freidlin-Wentzell's result [4] for the diffusion $dX_t^\varepsilon = a(X_t^\varepsilon)dt + \varepsilon b(X_t^\varepsilon)\varepsilon dW_t$ (W_t is Wiener process) holds. Namely,

$$J_\infty(\bar{u}) = \begin{cases} \frac{1}{2} \sum_{k=1}^{\infty} \frac{[u_k - a(u_{k-1})]^2}{b^2(u_{k-1})}, & u_0 = x_0 \\ \infty, & \text{otherwise.} \end{cases}$$

3. The next result plays an important role in proving Theorem 2.2, it also has an independent interest. Notation P_{x_0} will be used for designating ' $X_0^\varepsilon = x_0$ '.

We consider the model

$$X_k^\varepsilon = aX_{k-1}^\varepsilon + \varepsilon\xi_k.$$

Theorem 3.1. *Let the assumptions of Theorem 2.2 be fulfilled and $M \geq 1$. Then,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P_0 \left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1 \right) = -\frac{1}{2 \sum_{k=0}^{M-1} a^{2k}}.$$

In particular, for $|a| < 1$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P_0 \left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1 \right) = -\frac{1}{2}(1 - a^2) \left[1 - a^{2M} \right].$$

Proof. The family $\{(X_k^\varepsilon)_{k \geq 1}\}_{\varepsilon \rightarrow 0}$ obeys the LDP with the rate of speed ε^2 and the rate function

$$J_\infty(\bar{u}) = \begin{cases} \frac{1}{2} \sum_{k=1}^{\infty} [u_k - au_{k-1}]^2, & u_0 = 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Denote by

$$\mathfrak{A}_M = \{\bar{u} : \exists k \leq M \text{ with } |u_k| \geq 1 \text{ and } u_{k+1} = au_k, \forall k \geq M\}.$$

Obviously, any sequence $(\bar{u}_n)_{n \geq 1}$ from \mathfrak{A}_M converging in the metric ρ has a limit $\bar{u}_o \in \mathfrak{A}_M$, that is, \mathfrak{A}_M is a closed set. Hence, due to the LDP,

$$\overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbf{P} \left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1 \right) \leq - \min_{\bar{u} \in \mathfrak{A}_M} J_\infty(\bar{u}).$$

If $\bar{u}' \in \mathfrak{A}_M$, then $u'_{k+1} = au'_k$, $\forall k \geq M$, so that, $J_\infty(\bar{u}') = J_M(\bar{u}')$. If $\bar{u}' \in \mathfrak{A}_M$, then, there exist a number $\tau' = \inf\{1 \leq k \leq M : |u'_k| \geq 1\}$. Since $J_{\tau'}(\bar{u}') \leq J_M(\bar{u}')$ we may restrict a minimization procedure up to

$$\min_{\substack{u'_0=0, |u'_{\tau'}| \geq 1 \\ |u'_k| < 1, k \leq \tau'-1 \\ \tau' \leq M}} J_{\tau'}(\bar{u}') = \min_{\substack{u'_0=0, |u'_{\tau'}|=1 \\ |u'_k| < 1, k \leq \tau'-1 \\ \tau' \leq M}} J_{\tau'}(\bar{u}') \quad (3.1)$$

The proof of (3.1) is based on the fact that the lower bound for $\min_{\substack{u'_0=0, |u'_{\tau'}| \geq 1 \\ |u'_k| < 1, k \leq \tau'-1 \\ \tau' \leq M}} J_{\tau'}(\bar{u}')$

is attainable on $\min_{\substack{u'_0=0, |u'_{\tau'}|=1 \\ |u'_k| < 1, k \leq \tau'-1 \\ \tau' \leq M}}$.

To this end, by letting

$$u'_k = au'_{k-1} + w_k, \quad u_0 = 0, \quad k \leq \tau',$$

we find that $u'_{\tau'} = \sum_{k=1}^{\tau'} a^{\tau'-k} w_k$ and by $|u_{\tau'}| \geq 1$ and the Cauchy-Schwartz inequality $1 \leq \sqrt{\sum_{k=1}^{\tau'} a^{2(\tau'-k)} \sum_{k=1}^{\tau'} w_k^2}$. In other words,

$$\sum_{k=1}^{\tau'} w_k^2 \geq \frac{1}{\sum_{k=1}^{\tau'} a^{2(\tau'-k)}} \geq \frac{1}{\sum_{k=1}^M a^{2(M-k)}} = \frac{1}{\sum_{k=0}^{M-1} a^{2k}},$$

where the equality is attainable for $w_k \equiv Ka^{M-k}$ with free parameter K is chosen such that to keep $|u'_{\tau'}| = 1$ or $|u'_{\tau'}| \geq 1$ respectively. Thus, both sides of (3.1) possesses the same (attainable) lower bound: $\frac{1}{2 \sum_{k=0}^{M-1} a^{2k}}$ and, due to the LDP,

$$\overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbf{P} \left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1 \right) \leq - \frac{1}{2 \sum_{k=0}^{M-1} a^{2k}}.$$

In order to prove the lower bound

$$\underline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbf{P} \left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1 \right) \geq - \frac{1}{2 \sum_{k=0}^{M-1} a^{2k}},$$

we introduce an open subset of \mathfrak{A}_M :

$$\mathfrak{A}_M^o = \{\bar{u} : \exists k \leq M \text{ with } |u_k| > 1 \text{ and } u_{k+1} = au_k, \quad k \geq M\}.$$

So, due to the LDP,

$$\underline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbf{P} \left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1 \right) \geq - \min_{\bar{u} \in \mathfrak{A}_M^o} J_\infty(\bar{u}).$$

As previously, $\min_{\bar{u} \in \mathfrak{A}_M^o} J_\infty(\bar{u}) = \min_{\bar{u} \in \mathfrak{A}_M^o} J_M(\bar{u})$ and, moreover,

$$\min_{\bar{u} \in \mathfrak{A}_M^o} J_M(\bar{u}) = \frac{1}{2} \min_{\substack{u_0=0, |u_\tau| > 1 \\ |u_k| < 1, k \leq \tau-1 \\ \tau \leq M}} \sum_{k=1}^{\tau} (u_k - au_{k-1})^2 = \min_{\substack{u'_0=0, |u'_{\tau'}|=1 \\ |u'_k| < 1, k \leq \tau'-1 \\ \tau' \leq M}} J_{\tau'}(\bar{u}').$$

So, it is left to notice that for $|a| < 1$,

$$\sum_{k=0}^{M-1} a^{2k} = \frac{1}{1-a^2} [1 - a^{2M}].$$

□

4. LDP for $\varepsilon\xi$

In this Section, we prove statement 1) of Theorem 2.1.

There are different approaches for proving the LDP (see e.g. [1], [2], [9]). In our case, following Puhalskii's main theorem, ([14], see also Theorem 1.3 in [10]), it suffices to prove the exponential tightness:

$$\lim_{c \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(|\varepsilon\xi| \geq c) = -\infty, \quad (4.1)$$

and the local LDP: for every $v \in \mathbb{R}$

$$\overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(|\varepsilon\xi - v| \leq \delta) = \underline{\lim}_{\delta \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(|\varepsilon\xi - v| \leq \delta) = -I(v).$$

(4.1) is equivalent to

$$\lim_{c \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(\pm \varepsilon\xi \geq c) = -\infty.$$

By the Chernoff inequality $\mathbf{P}(\varepsilon\xi > c) \leq \exp(-tc/\varepsilon + H(t))$ and due to

$$\sup_{t>0} [t(c/\varepsilon) - H(t)] = \sup_{t \in \mathbb{R}} [t(c/\varepsilon) - H(t)]$$

and **(A.2)** we find that $\overline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(\varepsilon\xi > c) \leq -I(c) \xrightarrow{c \rightarrow \infty} -\infty$. The proof of $\underline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(-\varepsilon\xi > c) \leq -I(c) \xrightarrow{c \rightarrow \infty} -\infty$ is similar.

The local LDP is proved in two steps:

- 1) $\overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(|\varepsilon\xi - v| \leq \delta) \leq I(v)$
- 2) $\underline{\lim}_{\delta \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} q(\varepsilon) \log \mathbf{P}(|\varepsilon\xi - v| \leq \delta) \geq -I(v)$.

Set $Z = \exp(t\xi - H(t))$. Since $\mathbf{E}Z = 1$, for the proof of 1) we apply an obvious inequality $\mathbf{E}I(|\xi - u/\varepsilon| \leq \delta/\varepsilon)Z \leq 1$ which remains valid with Z is replaced by its lower bound $\underline{Z} = \exp(-(\delta/\varepsilon)|t| + t(u/\varepsilon) - H(t))$ on the set $\{|\xi - u/\varepsilon| \leq \delta/\varepsilon\}$. The latter provides

$$q(\varepsilon) \log \mathbf{P}(|\xi - v/\varepsilon| \leq \delta/\varepsilon) \leq q(\varepsilon)(\delta/\varepsilon)|t| - q(\varepsilon)[t(v/\varepsilon) - H(t)].$$

and, due to **(A.2)** and **(A.3)**, 1) holds.

For 2), it suffices to check only the validity of 2) for v with $I(v) < \infty$.

Let us denote $P(y)$ the distribution function of ξ . Set $\Lambda_t(y) = \exp(ty - H(t))$. Since $\int_{\mathbb{R}} \Lambda_t(y) dP(y) = 1$ one can introduce new distribution function $Q_t(y)$ which obeys the following properties:

$$\int_{\mathbb{R}} y dQ_t(y) = H'(t) \quad \text{and} \quad \int_{\mathbb{R}} [y - H'(t)]^2 dQ_t(y) = H''(t). \quad (4.2)$$

Since $I(v) < \infty$, t_v^ε is a number. Then, by taking $t = t_v^\varepsilon$, we find that

$$\begin{aligned} P(|\varepsilon\xi - v| \leq \delta) &= \int_{|y - (v/\varepsilon)| \leq (\delta/\varepsilon)} \exp(-t_v^\varepsilon y + H(t_u^\varepsilon)) dQ_{t_v^\varepsilon}(x) \\ &\geq \exp(-|t_v^\varepsilon|(\delta/\varepsilon) - t_v^\varepsilon v + H(t_v^\varepsilon)) \\ &\quad \times \int_{|x - (v/\varepsilon)| \leq (\delta/\varepsilon)} dQ_{t_v^\varepsilon}(x). \end{aligned}$$

Hence, 2) holds if, for instance,

$$\lim_{\varepsilon \rightarrow 0} \int_{|x - (v/\varepsilon)| \leq (\delta/\varepsilon)} dQ_{t_v^\varepsilon}(x) = 1, \quad \delta > 0,$$

while the latter is verified with the help of Chebyshev's inequality, (4.2) and **(A.3)**:

$$\begin{aligned} \int_{|x - (v/\varepsilon)| \leq (\delta/\varepsilon)} dQ_{t_v^\varepsilon}(x) &= 1 - \int_{|x - (u/\varepsilon)| > (\delta/\varepsilon)} dQ_{t_v^\varepsilon}(x) \\ &\geq 1 - \frac{\varepsilon^2}{\delta^2} \int_R (x - u/\varepsilon)^2 dQ_{t_v^\varepsilon}(x) \\ &= 1 - \frac{\varepsilon^2}{\delta^2} H''(t_v^\varepsilon) \\ &\rightarrow 1, \quad \varepsilon \rightarrow 0. \end{aligned}$$

5. LDP for $(\varepsilon\xi_k)_{k \geq 1}$

For $n > 1$, the LDP for the family $(\varepsilon\xi_k)_{1 \leq k \leq n}$ in the metric space (\mathbb{R}^n, ρ^n) , where for $x, y \in \mathbb{R}^n$ $\rho^n(x, y) = \sum_{k=1}^n |x_k - y_k|$ with the rate of speed $q(\varepsilon)$ and the rate function

$$I_n(v^n) = \sum_{k=1}^n I(v_k)$$

holds due to the vector $(\varepsilon\xi_k)_{1 \leq k \leq n}$ has i.i.d. entries (see [12]). Next, by Dawson-Gärtner's theorem (see [13] or [1]), the LDP for family $(\varepsilon\xi_k)_{k \geq 1}$ holds with the same rate of speed and the rate function

$$I(\bar{v}) = \sum_{k=1}^{\infty} I(v_k).$$

6. LDP for $(X_k^\varepsilon)_{k \geq m}$

The mapping $(\varepsilon\xi_k)_{k \geq m} \rightarrow (X_k^\varepsilon)_{k \geq m}$ is continuous in the metric ρ . Therefore, by the contraction principle (continuous mapping method) (see [3] and [11]) the family $(X_k^\varepsilon)_{k \geq m}$ obeys the LDP with the same rate of speed and the rate function

$$J_\infty(\bar{u}) = \inf_{(v_k, k \geq m: u_k = f(u_{k-1}, \dots, u_{k-m}, v_k))} I_\infty(\bar{v}),$$

where $\inf\{\emptyset\} = \infty$ and $I_\infty(\bar{v})$ is defined in (2.1) and

$$I_\infty(\bar{v}) = \sum_{k=m}^{\infty} \inf_{(v_k, k \geq m: u_k = f(u_{k-1}, \dots, u_{k-m}, v_k))} I(v_k).$$

Remark 1 holds true since by (2.2) the random variable ξ_1^{ii} is exponentially negligible with respect to the norming factor $q^i(\varepsilon)$: for any $\delta > 0$

$$\lim_{\varepsilon \rightarrow 0} q^i(\varepsilon) \log \mathbf{P}(|\varepsilon \xi_1^{ii}| > \delta) = -\infty.$$

7. Asymptotics of exit time.

In this Section we prove Theorem 2.2

Let M be an integer. It is clear that $\varepsilon^2 \log \mathbf{E} \tau^\varepsilon$ and $\varepsilon^2 \log \mathbf{E} \frac{\tau^\varepsilon}{M}$ have the same asymptotics as $\varepsilon \rightarrow 0$.

By taking into account $[z] \leq z \leq [z] + 1$, where $[z]$ is integer part of z , write

$$\begin{aligned} \mathbf{E} \frac{\tau^\varepsilon}{M} &\leq \mathbf{E} \left[\frac{\tau^\varepsilon}{M} \right] + 1 \\ &= \sum_{n=1}^{\infty} \mathbf{P} \left(\left[\frac{\tau^\varepsilon}{M} \right] \geq n \right) + 1 \\ &\leq \sum_{n=1}^{\infty} \mathbf{P}(\tau^\varepsilon \geq Mn) + 1 \\ &\leq \sum_{n=0}^{\infty} \mathbf{P}(\tau^\varepsilon > Mn) + 1. \end{aligned}$$

This upper bound implies the desired statement if

$$\lim_{M \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log \sum_{n=0}^{\infty} \mathbf{P}(\tau > Mn) \leq \frac{1}{2}(1 - a^2).$$

In order to establish the above upper bound, we use and an obvious equality

$$\mathbf{P}(\tau^\varepsilon > Mn) = \mathbf{P} \left(\max_{1 \leq k \leq Mn} |X_k^\varepsilon| < 1 \right)$$

and the Markov property of $(X_k^\varepsilon)_{k \geq 1}$:

$$\begin{aligned} &\mathbf{P} \left(\max_{1 \leq k \leq Mn} |X_k^\varepsilon| < 1 \right) \\ &= \mathbf{E} \left\{ I \left(\max_{1 \leq k \leq M(n-1)} |X_k^\varepsilon| < 1 \right) \mathbf{P}_{X_{M(n-1)}^\varepsilon} \left(\max_{M(n-1) < k \leq Mn} |X_k^\varepsilon| < 1 \right) \right\}. \end{aligned}$$

The time-homogeneity of X_k^ε implies

$$\begin{aligned} &I_{\{|X_{M(n-1)}^\varepsilon| < 1\}} \mathbf{P}_{X_{M(n-1)}^\varepsilon} \left(\max_{M(n-1) < k \leq Mn} |X_k^\varepsilon| < 1 \right) \\ &\leq I_{\{|X_{M(n-1)}^\varepsilon| < 1\}} \sup_{|x| < 1} \mathbf{P}_x \left(\sup_{0 < k \leq M} |X_k^\varepsilon| < 1 \right) \\ &\leq I_{\{|X_{M(n-1)}^\varepsilon| < 1\}} \left(1 - \inf_{|x| < 1} \mathbf{P}_x \left(\sup_{0 < k \leq M} |X_k^\varepsilon| \geq 1 \right) \right) \\ &= I_{\{|X_{M(n-1)}^\varepsilon| < 1\}} \left(1 - \mathbf{P}_0 \left(\sup_{0 < k \leq M} |X_k^\varepsilon| \geq 1 \right) \right), \end{aligned}$$

where the latter equality is provided by zero mean Gaussian distribution of X_k^ε , $k = 1, \dots, M$.

Hence we obtain the recurrence relation

$$\mathbf{P}(\tau^\varepsilon > Mn) \leq \mathbf{P}(\tau^\varepsilon > M(n-1)) \left(1 - \mathbf{P}_0 \left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1 \right) \right)$$

with $P(\tau^\varepsilon > 0) = 1$. Iterating it, we find that for any $n \geq 1$,

$$P(\tau^\varepsilon > Mn) \leq \left(1 - P_0\left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1\right)\right)^n.$$

Thus,

$$\sum_{n=0}^{\infty} P(\tau^\varepsilon > Mn) \leq \frac{1}{P_0\left(\max_{1 \leq k \leq M} |X_k^\varepsilon| \geq 1\right)}$$

and it is left to apply Theorem 3.1 \square

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